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Quasi-periodic solution of a new (2 + 1)-dimensional coupled soliton equation

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Abstract
A new $(2+1)$-dimensional integrable soliton equation is proposed, which has a close connection with the Levi soliton hierarchy. Through the nonlinearization of the Levi eigenvalue problems, we obtain a finite-dimensional integrable system. The Abel–Jacobi coordinates are constructed to straighten out the Hamiltonian flows, by which the solutions of both the 1 + 1 and 2 + 1 Levi equations are obtained through linear superpositions. An inversion procedure gives the quasi-periodic solution in the original coordinates in terms of the Riemann theta functions.

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1. Introduction

The study of finite-dimensional integrable systems (FDIS) and explicit solutions for various soliton equations has been very important in modern mathematics with ramifications to several areas of mathematics, physics and other sciences [1, 5, 10, 23]. Every finite-dimensional integrable system obtained would be looked at as a miracle (such as the Kovakevski top, geodesic flows on an ellipsoid, the harmonic oscillator equation on a sphere, the Calogero–Moser system, etc [11–18]. There are special relations between soliton equations and finite-dimensional integrable system [8, 10, 20–22, 25, 26]. Several methods have been employed to obtain explicit solutions of soliton equations, for instance, the inverse scattering transformation (IST), the algebra–geometric method, the Hirota bilinear method, the Lie symmetry method, etc [25, 30, 31]. Olver and Sokolov [32] give a method to obtain the classification, symmetries and Hamiltonian structures of integrable evolution equations which include some new important integrable systems. We try to use this method on equations (2.10) and (2.11) in our paper, but it is difficult because the u_{xx}, v_{xx} are alternate (compare with (4.7)–(4.25) in [32]). How one should use the above method to treat equation (2.10) is an open problem that we find interesting.

Recently, the nonlinearization approach of Lax pairs (or constrained flows) has been successfully applied to obtain finite-dimensional integrable systems from Lax pairs of soliton equations [1–4, 7, 9, 10]. There are many FDIS obtained by means of this approach. Very recently, the nonlinearization approach has been applied to obtain quasi-periodic solutions of soliton equations and was generalized to investigate soliton equations in two spatial and one temporal (i.e. (2 + 1)) dimensions [5, 6], in which two 2 + 1 integrable models, the special (2 + 1)-dimensional Toda equation and the well known Kadomtsev–Petviashvili equation, are associated with a same eigenvalue problem and some explicit solutions expressed by the theta function are obtained through Abel–Jacobi–Riemann inversion.

In this paper we consider a new (2 + 1)-dimensional coupled soliton equation

$$\begin{aligned} u_t &= \left(\tfrac{1}{4}u_{xx} - \tfrac{1}{2}u^3 - \tfrac{3}{2}uv^2 - 3v\partial^{-1}u_y\right)_x \\ v_t &= \left(\tfrac{1}{4}v_{xx} - 7v^3 - 3u^2v + 3v\partial^{-1}v_y\right)_x \end{aligned} \tag{1.1}$$

which is connected with the Levi hierarchy [19] and is similiar to the (2 + 1)-dimensional coupled mKdV equation that is important in physics and soliton theory. By using a map σ_λ: $\mathbb{C}^3 \mapsto sl(2, \mathbb{C})$, a 3×3 matrix differential Lenard operator and soliton hierarchy with their Lax pairs are deduced easily. Then the Bargmann constraint is obtained in a natural way by using another map $\tau_\lambda : \mathbb{C}^3 \mapsto \mathbb{C}^3$, through which the Lax pair of the soliton equation is nonlinearized into an N-dimensional Hamiltonian system. The conserved integrals $\{F_m\}$ are obtained by resorting to the generating function $\mathcal{F}(\lambda)$. It is shown that the generating function approach is powerful for proving involutivity and N dependence of the conserved integrals $\{F_m\}$. Hence the N-dimensional Hamiltonian system is integrable in the Liouville sense. By introducing the elliptic coordinates on the invariant torus of the N-dimensional Hamiltonian system, the Hamiltonian flows are mapped into linear flows on Abelian varieties and is integrated directly. Here the generating function approach plays a central role in the straightening of the flows, where the evolution of all the F_m-flow is obtained simultaneously through calculation of the evolution of the $\mathcal{F}(\lambda)$-flow on the Abelian varieties. Finally, the quasi-periodic solutions of the 2 + 1 coupled soliton equation are obtained by means of the Riemann theta functions.

2. Preliminaries

Consider the Levi spectral problem [19]:

$$\varphi_x = U\varphi \qquad U = \begin{pmatrix} \lambda + u & 2\lambda(v - u) \\ 1 & -\lambda - u \end{pmatrix}. \tag{2.1}$$

Define a linear map $\sigma_\lambda : C^3 \mapsto sl(2, \mathbb{C})$:

$$\sigma_\lambda(\alpha) = \begin{pmatrix} \alpha_1 + \lambda\alpha_3 & 2\lambda(\alpha_2 - \alpha_1) \\ \alpha_3 & -\alpha_1 - \lambda\alpha_3 \end{pmatrix} \qquad \alpha \in \mathbb{C}^3. \tag{2.2}$$

Here $U = \sigma_\lambda((u, v, 1)^T)$. Let $V = \sigma_\lambda(G)$, $G \in \mathbb{C}^3$. Through a direct calculation, we have

$$V_x - [U, V] = \sigma_\lambda\{(K - \lambda J)G\} \tag{2.3}$$

where $\boldsymbol{K}$, $\boldsymbol{J}$ are Lenard operator pairs:

$$K = \begin{pmatrix} \partial & 0 & 0 \\ 2v & \partial - 2u & 0 \\ -2 & 0 & \partial + 2u \end{pmatrix}$$
$$J = \begin{pmatrix} 0 & -2 & 2v \\ -2 & 0 & 2u \\ 0 & 0 & 0 \end{pmatrix} \qquad \partial \equiv \frac{\mathrm{d}}{\mathrm{d}x}. \tag{2.4}$$

Let $G = \sum_{j=0}^{\infty} \lambda^{-j} g_{j-1}$, $g_j = (g_j^1, g_j^2, g_j^3)^T \in \mathbb{C}^3$, it is easy to prove the following proposition:

Proposition 2.1. *The matrix* $V = \sigma_\lambda(G_\lambda)$ *satisfies the Lax equation* $V_x - [U, V] = 0$ *if and only if* $Kg_j = Jg_{j+1}$, $Jg_{-1} = 0$, $j = -1, 0, 1, \ldots$.

The Lenard gradients $\{g_j\}$ and Levi vector fields $\{X_j\}$ can been defined recursively by

$$\begin{aligned} &Kg_j = Jg_{j+1} && Jg_{-1} = 0 \\ &X_j = Jg_j = Kg_{j-1} && j = -1, 0, 1, 2, \ldots. \end{aligned} \tag{2.5}$$

The explicit recursive formula is

$$\begin{aligned} &g_{j+1,x}^3 = -2vg_j^1 + (2u - \partial)g_j^2 \\ &g_{j+1}^1 = \tfrac{1}{2}(\partial + 2u)g_{j+1}^3 \qquad j = 1, 2, \ldots \\ &g_{j+1}^2 = vg_{j+1}^3 - \tfrac{1}{2}g_{jx}^1. \end{aligned} \tag{2.6}$$

The first few members are:

$$g_{-1} = (u, v, 1)^T \qquad g_0 = (-\tfrac{1}{2}v_x - uv, -\tfrac{1}{2}u_x - v^2, -v)^T$$

$$g_1 = (\tfrac{1}{4}u_{xx} + \tfrac{3}{2}vv_x + \tfrac{3}{2}uv^2 - \tfrac{1}{2}u^3, \tfrac{1}{4}v_{xx} + vu_x + \tfrac{1}{2}uv_x + \tfrac{3}{2}v^3 - \tfrac{1}{2}u^2v, \tfrac{1}{2}u_x - \tfrac{1}{2}u^2 + \tfrac{3}{2}v^2)^T.$$

The corresponding vector fields are

$$\begin{aligned} &X_0 = (u_x, v_x, 0)^T \qquad X_1 = (-\tfrac{1}{2}v_{xx} - (uv)_x, \tfrac{1}{2}u_{xx} + uu_x - 3vv_x, 0)^T \\ &X_2 = (\tfrac{1}{4}u_{xx} + \tfrac{3}{2}vv_x + \tfrac{3}{2}v^2u - \tfrac{1}{2}u^3, \tfrac{1}{4}v_{xx} + \tfrac{3}{2}vu_x + \tfrac{5}{2}v^3 - \tfrac{3}{2}u^2v, 0)_x^T. \end{aligned} \tag{2.7}$$

Let

$$G_N = (\lambda^N G)_+ = \sum_{j=1}^{N} \lambda^{N-j} g_{j-1} \qquad V_N = \sigma_\lambda(G_N).$$

Then the Levi hierarchy is obtained from the zero-curvature form:

$$\frac{\partial}{\partial t_N}\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = X_N \quad \Longleftrightarrow \quad U_{t_N} - V_{Nx} + [U, V_N] = 0. \tag{2.8}$$

the Lax pairs of which are

$$\varphi_x = U\varphi \qquad \varphi_{t_N} = V_N\varphi. \tag{2.9}$$

The first two members are ($t_1 = y$, $t_2 = t$):

$$\text{Levi I:} \quad \begin{cases} u_y = -\tfrac{1}{2}v_{xx} - (uv)_x \\ v_y = \tfrac{1}{2}u_{xx} + uu_x - 3vv_x \end{cases} \tag{2.10}$$

$$\text{Levi II:} \quad \begin{cases} u_t = \tfrac{1}{4}u_{xxx} + \tfrac{3}{2}vv_{xx} + \tfrac{3}{2}v_x^2 + \tfrac{3}{2}v^2u_x - \tfrac{3}{2}u^2u_x + 3uvv_x \\ v_t = \tfrac{1}{4}v_{xxx} + \tfrac{3}{2}vu_{xx} + \tfrac{15}{2}v^2v_x + \tfrac{3}{2}v_xu_x - \tfrac{3}{2}u^2v_x - 3uvu_x. \end{cases} \tag{2.11}$$

The Lax pairs of (2.10) are

$$\varphi_x = U\varphi \qquad \varphi_y = V_1\varphi \tag{2.12}$$

$$V_1 = \begin{pmatrix} \lambda^2 + \lambda(u-v) - \frac{1}{2}v_x - uv & \lambda(v-u)_x + 2\lambda(\lambda - v)(v-u) \\ \lambda - v & -\lambda^2 - \lambda(u-v) + \frac{1}{2}v_x + uv \end{pmatrix}.$$

If $(u(x, y, t), v(x, y, t))$ is a compatible solution of (2.10) and (2.11), then $(u(x, y, t), v(x, y, t))$ is also a solution of the 2 + 1 coupled soliton equation

$$\begin{aligned} u_t &= (\tfrac{1}{4}u_{xx} - \tfrac{1}{2}u^3 - \tfrac{3}{2}uv^2 - 3v\partial^{-1}u_y)_x \\ v_t &= (\tfrac{1}{4}v_{xx} + 7v^3 - 3u^2v + 3v\partial^{-1}v_y)_x. \end{aligned} \tag{2.13}$$

3. The Levi–Bargmann system

Let $\varphi = (\varphi_1, \varphi_2)^T$ be a solution of (2.1), we define a map $\tau_\lambda : \mathbb{C}^2 \to \mathbb{C}^3$

$$\tau_\lambda(\varphi) = (-\lambda^2\varphi_2^2 + \lambda\varphi_1\varphi_2, -\tfrac{1}{2}\varphi_1^2 - \lambda^2\varphi_1^2 + \lambda\varphi_1\varphi_2, \lambda\varphi^2)^T. \tag{3.1}$$

It is easy to test the following formula:

$$K\tau_\lambda(\varphi) = \lambda J\tau_\lambda(\varphi). \tag{3.2}$$

Consider N copies of the linear Levi equation (2.1):

$$\partial_x \begin{pmatrix} p_j \\ q_j \end{pmatrix} = \begin{pmatrix} \alpha_j + u & 2\alpha_j(v-u) \\ 1 & -\alpha_j - u \end{pmatrix} \begin{pmatrix} p_j \\ q_j \end{pmatrix} \qquad j = 1, 2, \ldots, N \tag{3.3}$$

with distinct eigenvalues $\lambda = \alpha_1, \ldots, \alpha_N$.

Let

$$\tau_k \equiv (-\alpha_k q_k^2 + \alpha_k p_k q_k, \alpha_k p_k q_k - \alpha_k^2 q_k^2 - \tfrac{1}{2}p_k^2, \alpha_k q_k^2)^T \qquad k = 1, \ldots, N.$$

$$G_\lambda \equiv g_{-1} + \sum_{k=1}^{N} \frac{\tau_k}{\lambda - \alpha_k}.$$

Then we have a Lax matrix:

$$\begin{aligned} V(\lambda) &\equiv \sigma_\lambda(G_\lambda) = \triangle_\lambda + \sum_{k=1}^{N} \frac{\lambda\Gamma_k}{\lambda - \alpha_k} \\ &= \begin{pmatrix} \lambda + \lambda Q_\lambda(p, q) & -2\lambda\langle p, q\rangle - \lambda Q_\lambda(p, p) \\ 1 + Q_\lambda(q, \Lambda q) & -\lambda - \lambda Q_\lambda(p, q) \end{pmatrix} \equiv \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & -V_{11} \end{pmatrix} \end{aligned} \tag{3.4}$$

where $Q_\lambda(\xi, \eta) \equiv \sum_{j=1}^{N} \frac{\xi_j\eta_j}{\lambda - \alpha_j} = \sum_{s=0}^{\infty} \frac{\langle \xi, \Lambda^s \eta\rangle}{\lambda^{s+1}}$, $\Lambda \equiv \mathrm{diag}(\alpha_1, \ldots, \alpha_N)$, $p \equiv (p_1, \ldots, p_N)^T$,

$$\Gamma_k = \begin{pmatrix} p_k q_k & -p_K^2 \\ q_k^2 & -p_k q_k \end{pmatrix} \qquad \triangle\lambda = \begin{pmatrix} \lambda & -2\lambda\langle p, q\rangle \\ 1 - \langle q, q\rangle & -\lambda \end{pmatrix}. \tag{3.5}$$

Proposition 3.1. *The Lax matrix satisfies the following relation:*

$$V_x(\lambda) - [U, V(\lambda)] = \sigma_\lambda\left(Jg_0 - \sum_{k=1}^{N} \tau_k\right). \tag{3.6}$$

Proof. By using (2.3) and (3.2), we have

$$
\begin{aligned}
V_x(\lambda) - [U, V(\lambda)] &= \sigma_\lambda((K - \lambda J)G_\lambda) = \sigma_\lambda\left(Kg_{-1} + \sum_{k=1}^{N} \frac{(K - \lambda J)\tau_k}{\lambda - \alpha_k}\right) \\
&= \sigma_\lambda\left(Jg_0 + \sum_{k=1}^{N} \frac{(K - \alpha_k)\tau_k + (\alpha_k - \lambda)J\tau_k}{\lambda - \alpha_k}\right) \\
&= \sigma_\lambda\left(Jg_0 - \sum_{k=1}^{N} \tau_k\right).
\end{aligned}
$$

From here we obtain the Bargman constraint in a natural way:

$$
g_0 = \sum_{k=1}^{N} \tau_k. \tag{3.7}
$$

The explicit formula can be written in the following form by means of (3.3):

$$
\begin{aligned}
u &= \langle p, q\rangle - \langle q, \Lambda q\rangle \\
v &= -\langle q, \Lambda q\rangle.
\end{aligned} \tag{3.8}
$$

Then the spectral problem (2.1) is nonlinearized into a N-dimensional Hamiltonian system

$$
\begin{aligned}
p_x &= (\Lambda + \langle p, q\rangle - \langle q, \Lambda q\rangle)p - 2\langle p, q\rangle \Lambda q = -\frac{\partial H}{\partial q} \\
q_x &= p + (\langle q, \Lambda q\rangle - \Lambda - \langle p, q\rangle)q = \frac{\partial H}{\partial p}
\end{aligned} \tag{H}
$$

where

$$
H = \tfrac{1}{2}\langle p, p\rangle + \langle p, q\rangle\langle q, \Lambda q\rangle - \langle p, \Lambda q\rangle - \tfrac{1}{2}\langle p, q\rangle^2. \qquad \square
$$

Lemma 3.2. *Let $A, B \in sl(2, \mathbb{C})$, A satisfies the Lax equation $A_x = [A, B]$, then*

$$
\frac{\mathrm{d}}{\mathrm{d}x}(\det A) = 0.
$$

We noticed that $V(\lambda) = \sigma_\lambda(G_\lambda)$ is a solution of the Lax equation $V_x - [U, V] = 0$ in the Bargmann constraint, so $\mathcal{F}(\lambda) = \frac{1}{2\lambda} \det V(\lambda)$ is invariant along the x flow. Therefore, we have the generating function of integrals of equation (H):

$$
\mathcal{F}(\lambda) = \frac{1}{2\lambda} \det V(\lambda) = -\frac{\lambda}{2} + \sum_{m=0}^{\infty} \frac{F_m}{\lambda^{m+1}} \tag{3.9}
$$

where

$$
\begin{aligned}
F_0 &= \tfrac{1}{2}\langle p, p\rangle - \langle p, \Lambda q\rangle + \langle p, q\rangle\langle q, q\rangle - \tfrac{1}{2}\langle p, q\rangle^2 = H \\
F_m &= \tfrac{1}{2}(\langle p, \Lambda^m p\rangle - \langle p, q\rangle\langle p, \Lambda^m q\rangle) - \langle p, \Lambda^{m+1} q\rangle + \langle p, q\rangle\langle q, \Lambda^{m+1} q\rangle \\
&\quad + \tfrac{1}{2}\sum_{k=0}^{m-1} \begin{vmatrix} \langle p, \Lambda^k p\rangle & \langle p, \Lambda^k q\rangle \\ \langle p, \Lambda^{m-k} q\rangle & \langle q, \Lambda^{m-k} q\rangle \end{vmatrix}.
\end{aligned} \tag{3.10}
$$

Proposition 3.3. *$\{F_m\}$ are integrals of the Levi system (H); i.e. $\{F_m, H\} = \frac{\mathrm{d}F_m}{\mathrm{d}x} = 0$.*

Remark. The 'time' parts of Lax pairs for the Levi hierarchy are also nonlinearized into N-dimensional Hamiltonian systems by using (3.8):

$$p_{t_m} = -\frac{\partial F_m}{\partial q} \qquad q_{t_m} = \frac{\partial F_m}{\partial p}. \tag{F_m}$$

4. The integrability of the system (H)

4.1. The involutivity of $\{F_m\}$

Regard the generating function $\mathcal{F}(\lambda)$ as a Hamiltonian in the symplectic space $(\mathbb{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q)$. Let t_λ be the flow variable along the $\mathcal{F}(\lambda)$, then through a direct calculation, the canonical equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t_\lambda}\begin{pmatrix} p_k \\ q_k \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial \mathcal{F}(\lambda)}{\partial q_k} \\ \dfrac{\partial \mathcal{F}(\lambda)}{\partial p_k} \end{pmatrix} = W(\lambda, \alpha_k)\begin{pmatrix} p_k \\ q_k \end{pmatrix} \tag{4.1}$$

where

$$W(\lambda, \alpha_k) = \frac{V(\lambda)}{\lambda - \alpha_k} + V_0(\lambda) \qquad V_0(\lambda) = \begin{pmatrix} -V_{21} & -V_{12}/\lambda \\ 0 & V_{21} \end{pmatrix}. \tag{4.2}$$

Lemma 4.1. *The Γ_k satisfies the Lax equation:*

$$\frac{\mathrm{d}\Gamma_k}{\mathrm{d}t_\lambda} = [W(\lambda, \alpha_k), \Gamma_k]. \tag{4.3}$$

Proof. Let $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $\Gamma_k = \begin{pmatrix} p_k \\ q_k \end{pmatrix}(p_k\, q_k)\, S^T = PP^TS^T$, $P = \begin{pmatrix} p_k \\ q_k \end{pmatrix}$.

We note the fact:

$$SB^TS^T = -B \qquad \forall B \in sl(2, \mathbb{C}).$$

Therefore, by means of (4.1), we have

$$\begin{aligned} \frac{\mathrm{d}\Gamma_k}{\mathrm{d}t_\lambda} &= \begin{pmatrix} p_k \\ q_k \end{pmatrix}_{t_\lambda}(p_k\, q_k)\, S^T + \begin{pmatrix} p_k \\ q_k \end{pmatrix}(p_k\, q_k)_{t_\lambda}\, S^T \\ &= WPP^TS^T + PP^TW^TS^T \\ &= W\Gamma_k + \Gamma_k(-W) = [W, \Gamma_k]. \end{aligned}$$

□

Proposition 4.2. *The Lax matrix $V(\mu)$ satisfies the Lax equation along the $\mathcal{F}(\lambda)$ flow:*

$$\frac{\mathrm{d}}{\mathrm{d}t_\lambda} V(\mu) = [W(\lambda, \mu), V(\mu)] \qquad \forall \lambda, \mu \in \mathbb{C} \quad \lambda \neq \mu. \tag{4.4}$$

Proof. From equation (3.4) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t_\lambda}V(\mu) &= \sum_{k=1}^{N}\frac{\mu}{\mu-\alpha_k}\frac{\mathrm{d}\Gamma_k}{\mathrm{d}t_\lambda}+\frac{\mathrm{d}\triangle_\lambda}{\mathrm{d}t_\lambda}\\ &= \sum_{k=1}^{N}\frac{\mu}{\mu-\alpha_k}[W(\lambda,\alpha_k),\Gamma_k]+\frac{\mathrm{d}\triangle_\lambda}{\mathrm{d}t_\lambda}\end{aligned}$$

where

$$\begin{aligned}\sum_{k=1}^{N}\frac{\mu}{\mu-\alpha_k}[W(\lambda,\alpha_k),\Gamma_k] &= \sum_{k=1}^{N}\frac{\mu}{\mu-\alpha_k}\left[\frac{V(\lambda)}{\lambda-\alpha_k}+V_0(\lambda),\Gamma_k\right]\\ &= \sum_{k=1}^{N}\left(\frac{\mu}{\lambda-\mu}\left[V(\lambda),\left(\frac{1}{\mu-\alpha_k}-\frac{1}{\lambda-\alpha_k}\right)\mu\Gamma_k\right]+\frac{\mu}{\mu-\alpha_k}[V_0(\lambda),\Gamma_k]\right)\\ &= [W(\lambda,\mu),V(\mu)]+\left[\frac{V(\lambda)}{\lambda-\mu},\frac{\lambda}{\mu}\triangle_\lambda-\triangle_\mu\right]-\left[V_0(\lambda),\triangle_\mu\right].\end{aligned}$$

Through a direct calculation, the following formula can be proved:

$$\frac{\mathrm{d}\triangle_\lambda}{\mathrm{d}t_\lambda}+\left[\frac{V(\lambda)}{\lambda-\mu},\frac{\lambda}{\mu}\triangle_\lambda-\triangle_\mu\right]-\left[V_0(\lambda),\triangle_\mu\right]=0.$$ □

Theorem 4.3. $\{F_m\}$ *are involutive integrals of the Levi system* (H).

Proof. We note $V(\mu)$ is a solution of the Lax equation (4.4), $\mathcal{F}(\mu)=\frac{1}{2\mu}\det V(\mu)$. Therefore, $\mathcal{F}(\mu)$ is invariant along the t_λ flow, that is

$$\frac{\mathrm{d}}{\mathrm{d}t_\lambda}\mathcal{F}(\mu)=\{\mathcal{F}(\mu),\mathcal{F}(\lambda)\}=0.$$

On the other hand,

$$\{\mathcal{F}(\mu),\mathcal{F}(\lambda)\}=\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{1}{\lambda^{m+1}\mu^{n+1}}\{F_m,F_n\}.$$

Comparing the coefficients of λ, μ, we have $\{F_m,F_n\}=0$, $\forall m,n\in\mathbb{N}$. □

4.2. The independence of $\{F_m\}$

From (4.1) we can calculate

$$p_k\frac{\partial\mathcal{F}}{\partial p_k}-q_k\frac{\partial\mathcal{F}}{\partial q_k}=\frac{1}{\lambda-\alpha_k}\left(-2\alpha_k\langle p,q\rangle q_k^2+p_k^2+\mathrm{O}\left(\frac{1}{\lambda}\right)\right).$$

We define N-dimensional vectors with N distinct and sufficiently large constants $l_1,l_2,\ldots,l_N$:

$$\begin{aligned}&\vec{\mathcal{F}}=(\mathcal{F}(l_1),\ldots,\mathcal{F}(l_N))^T\\ &\vec{\eta}_k=\frac{\partial\vec{\mathcal{F}}}{\partial p_k}\qquad\vec{\zeta}_k=\frac{\partial\vec{\mathcal{F}}}{\partial q_k}\qquad\vec{\xi}=p_k\vec{\eta}_k-q_k\vec{\zeta}_k.\end{aligned}\tag{4.5}$$

If we omit the higher-order infinitely small $1/l_j$, there will be

$$\det|\vec{\xi}_1, \vec{\xi}_2, \ldots, \vec{\xi}_N| = \begin{vmatrix} \frac{1}{l_1-\alpha_1} & \cdots & \frac{1}{l_1-\alpha_N} \\ \cdots & \cdots & \cdots \\ \frac{1}{l_N-\alpha_1} & \cdots & \frac{1}{l_N-\alpha_N} \end{vmatrix} \prod_{k=1}^{N}(p_k^2 - 2\alpha_k\langle p, q\rangle q_k^2)$$

$$= \prod_{k=1}^{N}(p_k^2 - 2\alpha_k\langle p, q\rangle q_k^2) \prod_{j>k}(l_j - l_k) \prod_{\substack{j,k=1 \\ j\neq k}}^{N} (l_j - \alpha_k)^{-1} \prod_{j>k}(\alpha_j - \alpha_k).$$

Therefore, when $l_1, l_2, \ldots, l_N$ are distinct and sufficiently large, we shall have

$$\det|\vec{\xi}_1, \vec{\xi}_2, \ldots, \vec{\xi}_N| \neq 0$$

on the open set

$$\left\{(p, q)| \prod_{k=1}^{N}(p_k^2 - 2\langle p, q\rangle \alpha_k q_k^2) \neq 0\right\}.$$

which means that the vectors $\vec{\xi}_1, \vec{\xi}_2, \ldots, \vec{\xi}_N$ are linear independent,

$$N = \operatorname{rank}\{\vec{\xi}_1, \ldots, \vec{\xi}_N\} \leqslant \operatorname{rank}\{\vec{\eta}_1, \vec{\xi}_1, \ldots, \vec{\eta}_N, \vec{\xi}_N\} \leqslant N$$

$$\operatorname{rank} \frac{\partial \vec{\mathcal{F}}}{\partial(p, q)} = \operatorname{rank}\{\vec{\eta}_1, \vec{\zeta}_1, \ldots, \vec{\eta}_N, \vec{\zeta}\} = N$$

that is,

$$\operatorname{rank} \frac{\partial \vec{\mathcal{F}}}{\partial(p, q)} = N.$$

On the other hand,

$$\mathcal{F}(\lambda) = -\frac{\lambda}{2} + \sum_{m=0}^{\infty} \frac{F_m}{\lambda^{m+s}} = \frac{1}{2\lambda} \det V(\lambda)$$

$$= -\frac{\lambda}{2} + \frac{b_1\lambda^{N-1} + b_2\lambda^{N-2} + \cdots + b_N}{\lambda^N + a_1\lambda^{N-1} + \cdots + a_N}.$$

Comparing the coefficients of λ, we have

$$\vec{b} = A\vec{F} \qquad A = \begin{pmatrix} 1 & & & & & \\ a_1 & 1 & & & & \\ a_2 & a_1 & 1 & & & \\ \cdots & \cdots & \cdots & \cdots & & \\ a_{N-1} & a_{N-2} & a_{N-3} & \ldots & a_1 & 1 \end{pmatrix}$$

where

$$\vec{b} = (b_1, b_2 \ldots, b_N)^T \qquad \vec{F} = (F_1, F_2, \ldots, F_N)^T.$$

Therefore, $\operatorname{rank}\vec{b} = \operatorname{rank}\vec{F}$. Otherwise,

$$a(\lambda)\frac{\partial \mathcal{F}}{\partial p_k} = (\lambda^{N-1}, \lambda^{N-2}, \ldots, 1)\frac{\partial \vec{b}}{\partial p_k} \qquad a(\lambda)\frac{\partial \mathcal{F}}{\partial q_k} = (\lambda^{N-1}, \lambda^{N-2}, \ldots, 1)\frac{\partial \vec{b}}{\partial q_k}$$

that is,

$$\frac{\partial \vec{\mathcal{F}}}{\partial(p,q)} = \mathrm{diag}\left(\frac{1}{a(l_1)}, \ldots, \frac{1}{a(l_N)}\right) \begin{pmatrix} l_1^{N-1} & l_2^{N-2} & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots \\ l_1^{N-1} & l_2^{N-2} & \ldots & 1 \end{pmatrix} \frac{\partial \vec{b}}{\partial(p,q)}.$$

This implies

$$\mathrm{rank}\, \frac{\partial \vec{\mathcal{F}}}{\partial(p,q)} = \mathrm{rank}\, \frac{\partial \vec{b}}{\partial(p,q)} = N. \tag{4.6}$$

Hence $\vec{b} = A\vec{F}$ is functionally independent.

From the above discussion, we have found the following results.

Theorem 4.4. *On the open set* $\{(p,q)|\prod_{k=1}^{N}(p_k^2 - 2\alpha_k\langle p,q\rangle q_k^2) \neq 0\} \in \mathbb{R}^N$ $F_1, F_2, \ldots, F_N$ *are functionally independent.*

Proof. Because $\vec{b} = A\vec{F}$ in functionally independent and $\det(A) = 1$, $F_1, \ldots, F_N$ are functionally independent.

From theorems 4.3 and 4.4, $F_1, \ldots, F_N$ are N involutive and functionally independent integrals of (H). So the Hamiltonian system (H) is a complete integrable systems in the Liouville sense [23, 27]. □

Theorem 4.5. *The N-dimensional Hamiltonian system (H) is a complete integrable system in the Liouville sense. The N involutive pairs and functionally independent integrals are* $F_0, F_1, \ldots, F_{N-1}$.

5. Elliptic coordinates

It is easy to see that each of $\mathcal{F}(\lambda)$, $V_{12}(\lambda)$, $V_{21}(\lambda)$, as a rational function of λ, has simple poles at α_j, since the coefficient of $(\lambda - \alpha_j)^{-2}$ is zero in $\mathcal{F}(\lambda)$. We have

$$\begin{aligned} \mathcal{F}(\lambda) &= \frac{1}{2\lambda}\det V(\lambda) = -\frac{1}{2\lambda}(V_{11}^2 + V_{12}V_{21}) \equiv -\frac{1}{2}\frac{b(\lambda)}{a(\lambda)} \\ V_{12} &= -2\lambda\langle p,q\rangle - \lambda Q_\lambda(p,p) \equiv -2\lambda\langle p,q\rangle\frac{m(\lambda)}{a(\lambda)} \\ V_{21} &= 1 + Q_\lambda(q,\Lambda q) \equiv \frac{n(\lambda)}{a(\lambda)} \end{aligned} \tag{5.1}$$

where

$$m(\lambda) \equiv \prod_{k=1}^{N}(\lambda - \mu_k) \qquad n(\lambda) \equiv \prod_{k=1}^{N}(\lambda - \nu_k)$$

$$b(\lambda) \equiv \prod_{k=0}^{N+1}(\lambda - \beta_k) \qquad a(\lambda) \equiv \prod_{k=0}^{N}(\lambda - \alpha_k).$$

$\{\mu_k\}$ and $\{\nu_k\}$ are defined as *elliptic variables*. Expanding $V_{12}(\lambda)$ and $V_{21}(\lambda)$ in series of λ^{-k} and comparing the coefficients, we can obtain the following relations:

$$\begin{aligned}
\langle q, \Lambda q\rangle &= \sum_{k=1}^{N}(\alpha_k - \nu_k) \\
\langle p, p\rangle &= 2\langle p, q\rangle \sum_{k=1}^{N}(\alpha_k - \mu_k) \\
\langle q, \Lambda^2 q\rangle &= \frac{1}{2}\left\{\left[\sum_{k=1}^{N}(\alpha_k - \nu_k)\right]^2 + \sum_{k=1}^{N}(\alpha_k^2 - \nu_k^2)\right\} \\
\langle p, \Lambda p\rangle &= \langle p, q\rangle\left\{\left[\sum_{k=1}^{N}(\alpha_k - \mu_k)\right]^2 + \sum_{k=1}^{N}(\alpha_k^2 - \mu_k^2)\right\}
\end{aligned} \tag{5.2}$$

$$\sum_{k=1}^{N}\alpha_k = \sum_{j=1}^{N+1}\beta_j \qquad \sum_{k=1}^{N}\alpha_k^2 = \sum_{j=1}^{N+1}\beta_j^2 \tag{5.3}$$

$$\begin{aligned}
v &= -\langle q, \Lambda q\rangle = \sum_{k=1}^{N}(\nu_k - \alpha_k) \\
\frac{(u-v)_x}{u-v} &= \frac{1}{2}\frac{\langle p, p\rangle}{\langle p, q\rangle} - \langle q, \Lambda q\rangle = 2\sum_{k=1}^{N}(\nu_k - \mu_k).
\end{aligned} \tag{5.4}$$

Proposition 5.1. *The elliptic coordinates satisfy the evolution equations along the t_λ flow:*

$$\begin{aligned}
\frac{1}{2\sqrt{\mu_k R(\mu_k)}}\frac{\mathrm{d}\mu_k}{\mathrm{d}t_\lambda} &= -\frac{m(\lambda)}{a(\lambda)(\lambda - \mu_k)m'(\mu_k)} \\
\frac{1}{2\sqrt{\nu_k R(\nu_k)}}\frac{\mathrm{d}\nu_k}{\mathrm{d}t_\lambda} &= \frac{n(\lambda)}{a(\lambda)(\lambda - \nu_k)n'(\nu_k)}
\end{aligned} \tag{5.5}$$

where

$$R(\lambda) \equiv a(\lambda)b(\lambda) = \prod_{j=1}^{2N+1}(\lambda - \lambda_j)$$

with $\lambda_k = \alpha_k, \lambda_{N+j} = \beta_j$ $(k = 1, \ldots, N;\ j = 1, \ldots, N+1)$.

Proof. Substitute $\lambda = \mu_k, \nu_k$, respectively, in equation (5.1). We have

$$V_{11}(\mu_k) = \frac{\sqrt{\mu_k R(\mu_k)}}{a(\mu_k)} \qquad V_{11}(\nu_k) = \frac{\sqrt{\nu_k R(\nu_k)}}{a(\nu_k)}. \tag{5.6}$$

From the Lax equation $\frac{\mathrm{d}}{\mathrm{d}t_\lambda}V(\mu) = [W(\lambda, \mu), V(\mu)]$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t_\lambda}V_{12}(\mu) &= 2(W_{11}(\lambda, \mu)V_{12}(\mu) - W_{12}(\lambda, \mu)V_{11}(\mu)) \\
\frac{\mathrm{d}}{\mathrm{d}t_\lambda}V_{21}(\mu) &= 2(W_{21}(\lambda, \mu)V_{11}(\mu) - W_{11}(\lambda, \mu)V_{21}(\mu)).
\end{aligned}$$

Let $\mu = \mu_k$ and $\mu = \nu_k$, respectively. After some calculations we have equation (5.5). □

By means of the interpolation formula for polynomials with degree not more than $g = N$, we have ($j = 1, \ldots, g$)

$$\sum_{k=1}^{g} \frac{\mu_k^{g-j}}{2\sqrt{\mu_k R(\mu_k)}} \frac{\mathrm{d}\mu_k}{\mathrm{d}t_\lambda} = -\frac{\lambda^{g-j}}{a(\lambda)}$$
$$\sum_{k=1}^{g} \frac{\nu_k^{g-j}}{2\sqrt{\nu_k R(\nu_k)}} \frac{\mathrm{d}\nu_k}{\mathrm{d}t_\lambda} = \frac{\lambda^{g-j}}{a(\lambda)}. \tag{5.7}$$

Consider the hyperelliptic curve Γ:

$$\xi^2 - 4R(\lambda) = 0 \tag{5.8}$$

with genus $g = N$ since $\deg R(\lambda) = 2N + 1$. There is a linear independent holomophic differential [28] on Γ:

$$\tilde{\omega}_j = \frac{\lambda^{N-j}\,\mathrm{d}\lambda}{2\sqrt{\lambda R(\lambda)}} \qquad j = 1, 2, \ldots, N. \tag{5.9}$$

For a fixed λ_0, introduce the quasi-Abel–Jacobi coordinates:

$$\tilde{\phi}_j = \sum_{k=1}^{N} \int_{\lambda_0}^{\mu_k} \tilde{\omega}_j$$
$$\tilde{\psi}_j = \sum_{k=1}^{N} \int_{\lambda_0}^{\nu_k} \tilde{\omega}_j. \tag{5.10}$$

From (5.7), we have

Proposition 5.2 (Straightening of the $\mathcal{F}(\lambda)$ flow).

$$\frac{\mathrm{d}\tilde{\phi}_j}{\mathrm{d}t_\lambda} = -\frac{\lambda^{g-j}}{a(\lambda)} \qquad \frac{\mathrm{d}\tilde{\psi}_j}{\mathrm{d}t_\lambda} = \frac{\lambda^{g-j}}{a(\lambda)}. \tag{5.11}$$

Proposition 5.3 (Straightening of the F_k flow). *Let t_k be the variable of the F_k flow. Then*

$$\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}t_0} = 0 \qquad \frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}t_m} = -\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t_m} = -(A_m, A_{m-1}, \ldots, A_{m-g})^T \tag{5.12}$$

where $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_N)^T$, $t_0 = x$, $A_0 = 1$, $A_{-j} = 0$ ($j = 1, 2, \ldots$), A_j are the coefficients in the expansion

$$\frac{\lambda^N}{a(\lambda)} = \frac{1}{(1 - \alpha_1\lambda^{-1}) \cdots (1 - \alpha_N\lambda^{-1})} = 1 + \sum_{j=1}^{\infty} A_j \lambda^{-j} \tag{5.13}$$

which could be represented through the power sums of α_k, $\sigma_l = \sum_{k=1}^{N} \alpha_k^l$:

$$A_1 = \sigma_1 \qquad A_2 = \tfrac{1}{2}(\sigma_2 + \sigma_1^2) \qquad A_3 = \tfrac{1}{6}(2\sigma_3 + 3\sigma_2\sigma_1 + \sigma_1^3)$$

with the recursive formula

$$A_k = \frac{1}{k}\left(\sigma_k + \sum_{\substack{i+j=k \\ i,j \geqslant 1}} \sigma_j A_j\right). \tag{5.14}$$

Proof. According to the definition of the Poisson bracket:

$$\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}t_\lambda} = (\tilde{\phi}, \mathcal{F}_\lambda) = \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}} (\tilde{\phi}, F_k) = \frac{1}{\lambda^{k+1}} \frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}t_k}.$$

With the supplementary definition $A_0 = 1$, $A_{-j} = 0$ $(j = 1, 2, 3, \ldots)$, the comparison of the coefficients of λ^{-k-1} in equation (5.13) yields $\mathrm{d}\tilde{\phi}_j/\mathrm{d}t_k = A_{k-j}$, and

$$\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}t_m} = -\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}t_m} = -(A_m, A_{m-1}, \ldots, A_{m-g})^T.$$

The proof of equation (5.14) is elementary, resorting to the expansion $\sum_{k=1}^{\infty} k^{-1}\sigma_k\lambda^{-k}$ of the right-hand side of equation (5.13).

Now we consider the zeros on the hyperelliptic curve Γ with genus $g = N$ since $\deg R = 2N + 1$. At $\lambda = \infty$, the affine equation is transformed into ($z = \lambda^{-1/2}$, $\hat{\xi} = z^{2N+1}\xi$):

$$\hat{\xi} = 4R_*(z)$$

with

$$R_*(z) = z^{4N+2} R(z^{-2}) = \prod_{j=1}^{2N+1} (1 - \lambda_j z^2). \tag{5.15}$$

Take the canonical basis of cycles on $\Gamma : a_1, \ldots, a_g, b_1, \ldots, b_g$. Let $C = (C_{js})$ be the inverse of the periodic matrix (A_{sk}):

$$C = (A_{sk})^{-1}_{g\times g} \qquad A_{sk} = \int_{a_k} \tilde{\omega}_s. \tag{5.16}$$

Then for the normalized holomorphic differential

$$\omega_j = \sum_{s=1}^{g} C_{js}\tilde{\omega}_s \qquad \omega = (\omega_1, \ldots, \omega_g)^T = C\tilde{\omega} \tag{5.17}$$

we have

$$\int_{a_k} \omega_j = \delta_{jk} \qquad \int_{b_k} \omega_j = B_{jk}. \tag{5.18}$$

According to the Riemannian relation, the matrix $B = (_{jk})$ is symmetric and has a positive-definite imaginary part, and is used to construct the Riemannian theta function [29] of Γ:

$$\theta(\zeta) = \sum_{z\in\mathbb{Z}^g} \exp \pi\sqrt{-1}(\langle Bz, z\rangle + 2\langle \zeta, z\rangle) \qquad \zeta \in \mathbb{C}^g.$$

For fixed P_0 on Γ, the Abel–Jacobi coordinates are defined as

$$\begin{aligned} \phi &= \sum_{k=1}^{N} \int_{P_0}^{(\mu_k, \xi(\mu_k))} \omega \\ \psi &= \sum_{k=1}^{N} \int_{P_0}^{(\nu_k, \xi(\nu_k))} \omega. \end{aligned} \tag{5.19}$$

□

Lemma 5.4. *Let* $S_k = \lambda_1^k + \cdots + \lambda_{2N+1}^k$. *Then the coefficients of* $\frac{1}{\sqrt{z^2 R_*(z)}} = \sum_{k=0}^{\infty} R_k z^{2k}$ *satisfy the recursive formula*

$$R_0 = 1 \qquad R_1 = \frac{1}{2} S_1 \qquad R_k = \frac{1}{2k}\left(S_k + \sum_{\substack{i+j=k \\ i,j \geqslant 1}} S_j R_k \right). \tag{5.20}$$

Proof. Since $\ln(1-t) = -\sum_{k=1}^{\infty} k^{-1} t^k$, we have

$$\ln \frac{1}{\sqrt{z^2 R_*(z)}} = -\frac{1}{2} \sum_{j=1}^{2N+1} \ln(1 - \lambda_j z^2) = \sum_{k=1}^{\infty} S_k z^{2k}.$$

By differentiating with regard to z and comparing the coeffients of z, we obtain equation (5.20).

Let $C_1, \ldots, C_g$ be the column vectors of C defined by equation (5.16). Then by direct calculations, the coefficients in

$$\frac{1}{2\sqrt{z^2 R_*(z)}}(C_1 z + C_2 z^2 + \cdots + C_g z^g) = \sum_{k=1}^{\infty} \Omega_k z^{2k-1} \tag{5.21}$$

are

$$\Omega_k = \tfrac{1}{2}(R_{k-1} C_1 + \cdots + R_{k-g} C_g) \tag{5.22}$$

with the additional definition $R_{-s} = 0$ $(s = 1, 2, \ldots)$. Specifically,

$$\Omega_0 = 0 \qquad \Omega_1 = \tfrac{1}{2} C_1$$

$$\Omega_k = \tfrac{1}{2}(R_{k-1} C_1 + \cdots + R_1 C_{k-1} + C_k) \qquad (k = 1, \ldots, g). \qquad \square$$

Proposition 5.5. *The* t_k *flow is straightened by the Abel–Jacobi coordinates:*

$$\frac{\mathrm{d}\phi}{\mathrm{d}t_k} = -\Omega_k \qquad \frac{\mathrm{d}\psi}{\mathrm{d}t_k} = \Omega_k. \tag{5.23}$$

Proof. From equation (5.11) we obtain

$$\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}t_\lambda} = \frac{\lambda^g}{2\sqrt{\lambda R(\lambda)}}(\lambda^{-1}, \ldots, \lambda^{-g})$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t_\lambda} = C \frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}t_\lambda} = \frac{\lambda^g}{2\sqrt{\lambda R(\lambda)}}(C_1 \lambda^{-1} + \cdots + C_g \lambda^{-g}) = \sum_{k=1}^{\infty} \Omega_k \lambda^{-k-1}.$$

Hence we obtain the first part of equation (5.23) after comparing the coefficients of λ^{-k-1}, while the second part is obtained similarly.

The straightened equations (5.23) are easily integrated by quadratures: $\phi = \phi_0 - \sum \Omega_k t_k$. And the evolution picture of the confocal flow and Levi flow becomes very simple through the 'window' of the Abel–Jacobi coordinates ϕ (as well as ψ):

confocal F_k: $\quad \phi = \phi_0 - \Omega_k t_k$

Levi X_k: $\quad \phi = \phi_0 - \Omega_1 x - \Omega_k t_k$.

Specifically,

$$\begin{aligned} &\text{Levi–Bargmann stationary equation } (H): && \phi = \phi_0 - \Omega_1 x \\ &\text{Levi equation I:} && \phi = \phi_0 - \Omega_1 x - \Omega_2 y \\ &\text{Levi equation II:} && \phi = \phi_0 - \Omega_1 x - \Omega_2 t \\ &2+1 \text{ coupled equation:} && \phi = \phi_0 - \Omega_1 x - \Omega_2 y - \Omega_3 t. \end{aligned} \tag{5.24}$$

The corresponding explicit solutions are obtained by some inversion procedures from ϕ, ψ to the coordinates u, v via the elliptic coordinates $\{\mu_k, \nu_k\}$. □

6. Inversion from ϕ, ψ to $\{\mu_k\}$, $\{\nu_k\}$ and quasi-periodic solutions of LI, LII and (2.3)

The Abel–Jacobi map A: $\mathrm{Div}(\Gamma) \to \mathcal{J} = \mathbb{C}^g/\mathcal{T}$ is defined by

$$A(P) = \int_{P_0}^{P} \omega \qquad A\left(\sum n_k P_k\right) = \sum n_k A(P_k) \tag{6.1}$$

where $P_0 = \xi(\lambda_0)$ is fixed, $\mathrm{Div}(\Gamma)$ is the divisor group, and the lattice $\mathcal{T}$ is spanned by the periodic vector $\{\delta_j; B_j\}$, which are the column vectors of E and (B_{js}) defined by equation (5.18). The definition of Abel–Jacobi coordinates is rewritten as

$$\phi = A\left\{\sum_{j=1}^{g} \xi(\mu_j)\right\} \qquad \psi = A\left\{\sum_{j=1}^{g} \xi(\nu_j)\right\}. \tag{6.2}$$

According to the Riemann theorem [29], there exists a constant vector K such that:

(a) $\theta(A(P(\lambda)) - \phi - K)$ has exactly g zeros at $\lambda = \mu_1, \ldots, \mu_g$;
(b) $\theta(A(P(\lambda)) - \psi - K)$ has exactly g zeros at $\lambda = \nu_1, \ldots, \nu_g$.

And we have the inversion formulae:

$$\begin{aligned} \sum_{j=1}^{g} \mu_j^s &= I_s(\Gamma) - \operatorname*{Res}_{\lambda=\infty} \lambda^s \,\mathrm{d} \ln \theta(A(P(\lambda)) - \phi - K) \\ \sum_{j=1}^{g} \nu_j^s &= I_s(\Gamma) - \operatorname*{Res}_{\lambda=\infty} \lambda^s \,\mathrm{d} \ln \theta(A(P(\lambda)) + \psi + K) \end{aligned} \tag{6.3}$$

where

$$I_s(\Gamma) = \sum_{j=1}^{g} \int_{a_j} \lambda^s \omega_j$$

is independent with ϕ.

In the neighbourhood of $\lambda = \infty$,

$$\omega = C\tilde{\omega} = -(C_1 z^2 + \cdots + C_g z^{2g}) \frac{\mathrm{d}z}{\sqrt{z^2 R_*(z)}}.$$

With the help of equation (5.21) we obtain

$$A(\phi(z^{-1})) = -\eta + \sum_{k=1}^{\infty} \frac{1}{2k} \Omega_k z^{2k}$$

where

$$\eta = \int_{\infty}^{P_0} \omega. \tag{6.4}$$

Hence we have the power-series expansions near $\lambda = \infty$ in the local coordinate $z = \lambda^{-1/2}$:

$$\begin{aligned}
\ln\theta(A(\xi(\lambda)) - \phi - K) &= \ln\theta\left(-\phi - K - \eta - \sum_{k=1}^{\infty} \frac{1}{2k}\Omega_k z^{2k}\right) \\
&= \ln\theta(\phi + K + \eta) + \sum_{k=1}^{\infty} a_k z^{2k} \\
\ln\theta(A(\xi(\lambda)) - \psi - K) &= \ln\theta\left(\psi + K + \eta + \sum_{k=1}^{\infty} \frac{1}{2k}\Omega_k z^{2k}\right) \\
&= \ln\theta(-\phi - K - \eta) + \sum_{k=1}^{\infty} b_k z^{2k}.
\end{aligned} \tag{6.5}$$

Here the fact $\theta(\zeta) = \theta(-\zeta)$ is used. Equation (6.3) can be written as

$$\begin{aligned}
\sum_{j-1}^{g} \mu_j^s &= I_s(\Gamma) - s a_s \\
\sum_{j=1}^{g} \nu_j^s &= I_s(\Gamma) - s b_s.
\end{aligned} \tag{6.6}$$

The residue at $\lambda = \infty$ can be obtained

$$\operatorname*{Res}_{\lambda=\infty} \lambda \,\mathrm{d} \ln\theta = 2a_2 = \sum_{j=1}^{g}\sum_{k=1}^{g} \Omega_{j1}\Omega_{k1} \frac{\partial^2 \ln\theta}{\partial\zeta_j \partial\zeta_k}$$

$$\begin{aligned}
\operatorname*{Res}_{\lambda=\infty} \lambda^2 \,\mathrm{d} \ln\theta = 4a_4 = \frac{4}{3}\sum_{j=1}^{g}\sum_{k=1}^{g} \Omega_{j2}\Omega_{k1} \frac{\partial^2 \ln\theta}{\partial\zeta_j \partial\zeta_k} \\
+\frac{4}{3}\sum_{j=1}^{g}\sum_{k=1}^{g}\sum_{l=1}^{g}\sum_{m=1}^{g} \frac{\partial^4 \ln\theta}{\partial\zeta_j \partial\zeta_k \partial\zeta_l \partial\zeta_m} \Omega_{j1}\Omega_{k1}\Omega_{l1}\Omega_{m1}.
\end{aligned}$$

The special case is

$$\begin{aligned}
\sum_{j-1}^{g} \mu_j &= I_1(\Gamma) + \sum_{j=1}^{g} \Omega_{j1}\Omega_{k1} \frac{\partial^2 \ln\theta}{\partial\zeta_j \partial\zeta_k} \\
\sum_{j=1}^{g} \nu_j &= I_1(\Gamma) + \sum_{j=1}^{g} \Omega_{j1}\Omega_{k1} \frac{\partial^2 \ln\theta^*}{\partial\zeta_j \partial\zeta_k}
\end{aligned} \tag{6.7}$$

where

$$\theta = \theta(\phi - K - \eta) = \theta\left(\sum_k \Omega_k t_k - \phi_0 - K - \eta\right)$$

$$\theta^* = \theta(\psi + K + \eta) = \theta\left(\sum_k \Omega_k t_k + \psi_0 + K + \eta\right).$$

Denote $x = x_0$, $y = t_1$, $t = t_2$. By the chain rule of differentiation for composition functions, equation (6.7) is further simplified as

$$\sum_{j-1}^{g} \mu_j = I_1(\Gamma) - \frac{\partial^2 \ln\theta}{\partial x^2}$$
$$\sum_{j=1}^{g} \nu_j = I_1(\Gamma) + \frac{\partial^2 \ln\theta^*}{\partial x^2}. \tag{6.8}$$

Proposition 6.1. *The quasi-periodic solution of Levi I is*

$$\begin{aligned} v(x, y) &= \frac{\partial^2}{\partial x^2} \ln\theta(\Omega_1 x + \Omega_2 y + K + \eta) + N_1 \\ u(x, y) &= \exp\left\{2\frac{\partial}{\partial x} \ln(\theta(\Omega_1 x + \Omega_2 y + K + \eta)\theta(\Omega_1 x + \Omega_2 y - K - \eta))\right\} \\ &\quad + \frac{\partial^2}{\partial x^2} \ln\theta(\Omega_1 x + \Omega_2 y + K + \eta) + N_2 \end{aligned} \tag{6.9}$$

where N_1 and N_2 are constants.

Proof. From equations (5.4) and (6.8), we have

$$\begin{aligned} v = -\langle q, \Lambda q\rangle &= \sum_{k=1}^{N} (\nu_k - \alpha_k) \\ &= I_1(\Gamma) - \sum_{k=1}^{N} \alpha_k + \frac{\partial^2}{\partial x^2} \ln\theta(\Omega_1 x + \Omega_2 y + K + \eta) \\ &= N_1 + \frac{\partial^2}{\partial x^2} \ln\theta(\Omega_1 x + \Omega_2 y + K + \eta). \end{aligned}$$

From equation (5.4), we have

$$\begin{aligned} \frac{(u-v)_x}{u-v} &= \frac{1}{2}\frac{\langle p, p\rangle}{\langle p, q\rangle} - \langle q, \Lambda q\rangle = 2\sum_{k=1}^{N} (\nu_k - \mu_k) \\ &= 2\frac{\partial^2}{\partial x^2} \ln(\theta(\Omega_1 x + \Omega_2 y + K + \eta)\theta(\Omega_1 x + \Omega_2 y - K - \eta)). \end{aligned}$$

Equation (6.8) can be obtained if we omit the integral constant. □

We can also obtain the solution of Levi II in a similar way.

Proposition 6.2. *The quasi-periodic solution of Levi II is*

$$\begin{aligned} v(x, t) &= \frac{\partial^2}{\partial x^2} \ln\theta(\Omega_1 x + \Omega_3 t + K + \eta) + M_1 \\ u(x, t) &= \exp\left\{2\frac{\partial}{\partial x} \ln(\theta(\Omega_1 x + \Omega_3 t + K + \eta)\theta(\Omega_1 x + \Omega_3 t - K - \eta))\right\} \\ &\quad + \frac{\partial^2}{\partial x^2} \ln\theta(\Omega_1 x + \Omega_3 t - K - \eta) + M_2 \end{aligned} \tag{6.10}$$

where M_1 and M_2 are constants.

When $u(x, y, t)$, $v(x, y, t)$ is a compatible solution of Levi I and Levi II, Then $u(x, y, t)$, $v(x, y, t)$ is also a solution of the $2 + 1$ coupled soliton equation (1.1).

Theorem 6.3. *For the* $2+1$ *coupled equation*

$$u_t = \left(\tfrac{1}{4}u_{xx} - \tfrac{1}{2}u^3 2 - \tfrac{3}{2}uv^2 - 3v\partial^{-1}u_y\right)_x$$
$$v_t = \left(\tfrac{1}{4}v_{xx} - 7v^3 - 3u^2 v + 3v\partial^{-1}v_y\right)_x$$

there exists a quasi-periodic solution

$$v(x, y, t) = \frac{\partial^2}{\partial x^2}\ln\theta(\Omega_1 x + \Omega_2 y + \Omega_3 t + K + \eta) + C_1$$
$$u(x, y, t) = \exp\left\{2\frac{\partial}{\partial x}\ln(\theta(\Omega_1 x + \Omega_2 y + \Omega_3 t + K + \eta)\theta(\Omega_1 x + \Omega_2 y + \Omega_3 t - K - \eta))\right\} \quad (6.11)$$
$$+\frac{\partial^2}{\partial x^2}\ln\theta(\Omega_1 x + \Omega_2 y + \Omega_3 t + K + \eta) + C_2$$

where C_1, C_2 *are constants.*

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